Casimir effect: running Newton constant or cosmological term

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We argue that the instability of Euclidean Einstein gravity is an indication that the vacuum is non perturbative and contains a condensate of the metric tensor in a manner reminiscent of Yang-Mills theories. As a simple step toward the characterization of such a vacuum the value of the one-loop effective action is computed for Euclidean de Sitter spaces as a function of the curvature when the unstable conformal modes are held fixed. Two phases are found, one where the curvature is large and gravitons should be confined and another one which appears to be weakly coupled and tends to be flat. The induced cosmological constant is positive or negative in the strongly or weakly curved phase, respectively. The relevance of the Casimir effect in understanding the UV sensitivity of gravity is pointed out.

I. EFFECTIVE POTENTIAL FOR THE CURVATURE

A. Formal expressions

$$e^{W[j]} = \int D[g] e^{-S[g]+\int dx\sqrt{g} j^{\mu\nu} g_{\mu\nu}},$$

$$S[g] = -\kappa_B^2 \int dx\sqrt{g}(R - 2\lambda_B),$$

$$\Gamma[g] = -W[j] + \int dx\sqrt{g} j^{\mu\nu} g_{\mu\nu}$$

$$g = \frac{\delta W[j]}{\delta j}.$$

$$\gamma(R) = \Gamma[g^{(R)}]$$

$$g_{\mu\nu} \longrightarrow \Omega^2 g_{\mu\nu}, \quad R \longrightarrow \frac{R}{\Omega^2} - 6\frac{\Box\Omega}{\Omega^3},$$

The one-loop approximation yields

$$W[j] = -S[\bar{g}] + \int dx\sqrt{g} j^{\mu\nu} \bar{g}_{\mu\nu} + \frac{1}{2}\mathrm{T}r \ln \frac{\delta^2 S[\bar{g}]}{\delta g \delta g}$$

$$\frac{\delta^2 S[\bar{g}]}{\delta g \delta g} = j$$

$$\Gamma[g] = S[g] + \frac{1}{2}\mathrm{T}r \ln \frac{\delta^2 S[g]}{\delta g \delta g},$$

$$\gamma(R) = S[g^{(R)}] + \frac{1}{2}\mathrm{T}r \ln \frac{\delta^2 S[g^{(R)}]}{\delta g \delta g},$$

B. Gauge fixing and regularization

$$S_{gf}[g^{(R)},h]=\frac{1}{2}\int dx\sqrt{g^{(R)}}g^{(R)\mu\nu}f_\mu[g^{(R)},h]f_\nu[g^{(R)},h]$$

$$\begin{aligned}
f_\mu[g^{(R)},h] &= \sqrt{2}\kappa_B f_\mu^{\nu\rho}[g^{(R)}]h_{\nu\rho},\\
f_\mu^{\alpha\beta}[g^{(R)}] &= \frac{1}{2}\delta_\mu^\alpha g^{(R)\beta\gamma}D_\gamma+\frac{1}{2}\delta_\mu^\beta g^{(R)\alpha\gamma}D_\gamma-\frac{1}{2}g^{(R)\alpha\beta}D_\mu,
\end{aligned}$$

$g=\bar{g}+h$, $\kappa_B^2=1/16\pi G_B$.

$$e^{W[j]}=\int D[h]D[c]D[\bar{c}]e^{-S[g^{(R)}+h]-S_{gf}[g^{(R)},h]+S_{gh}[g^{(R)},h,c,\bar{c}]+\int dx\sqrt{g}j^{\mu\nu}(g^{(R)}_{\mu\nu}+h_{\mu\nu})}$$

$$S_{gh}[g^{(R)},h,c,\bar{c}]=-\sqrt{2}\int dx\sqrt{g^{(R)}}\bar{c}^\mu\mathcal{M}_{\mu\nu}[g^{(R)},h]c^\nu$$

$$\mathcal{M}_{\alpha\beta}=\frac{\delta f_\alpha[g^{(R)},h^\epsilon]}{\delta\epsilon^\beta}=\sqrt{2}\kappa_B f_\alpha^{\mu\nu}[g^{(R)}]\left(\frac{\partial g_{\mu\nu}}{\partial x^\beta}-\frac{\partial g_{\beta\nu}}{\partial x^\mu}-\frac{\partial g_{\mu\beta}}{\partial x^\nu}\right).$$

$$S_B[g]=S[g]+\int dx\sqrt{g}g_{\mu\nu}f_\Lambda(-D^2)g^{\mu\nu},$$

$$f_\Lambda(z)=\begin{cases}0 & z<\Lambda^2,\\ \infty & z>\Lambda^2.\end{cases}$$

$$\begin{aligned}
\gamma(R) &= S_B[g^{(R)}]+\gamma^{(1)}(R),\\
\gamma^{(1)}(R) &= \frac{\kappa_B^2}{2}\mathrm{Tr}\ln\mathcal{K}-\mathrm{Tr}\ln\mathcal{M},
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}_{(\mu\nu),(\rho\sigma)} &= \kappa_B^2\sqrt{g^{(R)}}\left(-Z_{(\mu\nu),(\rho\sigma)}D^2+U_{(\mu\nu),(\rho\sigma)}\right),\\
\mathcal{M}_{\mu\nu} &= -\sqrt{2}\kappa_B^2\sqrt{g^{(R)}}\left(g^{(R)}_{\mu\nu}D^2+R_{\mu\nu}\right),
\end{aligned}$$

$$Z_{(\mu\nu),(\rho\sigma)} = \frac{1}{4}\left(g^{(R)}_{\mu\rho} g^{(R)}_{\nu\sigma} + g^{(R)}_{\mu\sigma} g^{(R)}_{\nu\rho} - g^{(R)}_{\mu\nu} g^{(R)}_{\rho\sigma}\right),$$

$$\begin{aligned} U_{(\mu\nu),(\rho\sigma)} \ = \ & Z_{(\mu\nu),(\rho\sigma)}(R - 2\lambda_B) + \frac{1}{2}\left(g^{(R)}_{\mu\nu} R_{\sigma\rho} + R_{\mu\nu} g^{(R)}_{\sigma\rho}\right) \\ & -\frac{1}{4}\left(g^{(R)}_{\mu\rho} R_{\nu\sigma} + g^{(R)}_{\mu\sigma} R_{\nu\rho} + R_{\mu\rho} g^{(R)}_{\nu\sigma} + R_{\mu\sigma} g^{(R)}_{\nu\rho}\right) \\ & -\frac{1}{2}\left(R_{\mu\rho} R_{\nu\sigma} + R_{\mu\sigma} R_{\nu\rho}\right), \end{aligned}$$

C. Spin projection

$$h_{\mu\nu} = \tilde{h}^{TT}_{\mu\nu} + \tilde{h}^{LT}_{\mu\nu} + \tilde{h}^{LL}_{\mu\nu} + \tilde{h}^{Tr}_{\mu\nu}$$

$\tilde{h}^{Tr}_{\mu\nu} = g^{(R)}_{\mu\nu}\tilde{\phi}$, $\tilde{\phi} = g^{(R)}_{\mu\nu} h^{\mu\nu}$, $\tilde{h}^{LT}_{\mu\nu} = D_\mu \tilde{\xi}'_\nu + D_\nu \tilde{\xi}'_\mu$ and $\tilde{h}^{LL}_{\mu\nu} = D_\mu D_\nu \tilde{\sigma}' - \frac{1}{4} g^{(R)}_{\mu\nu} D^2 \tilde{\sigma}'$. $D_\mu \tilde{\xi}^\mu = 0$, $g^{(R)\mu\nu}\tilde{h}^{TT}_{\mu\nu} = 0$ and $D^\mu \tilde{h}^{TT}_{\mu\nu} = 0$.

$$\begin{aligned} \langle h_1 | h_2 \rangle \ &= \ \int dx \sqrt{g} h_{1;\mu\nu} h_2^{'\mu\nu} \\ &= \ \int dx \sqrt{g} \Big[\tilde{h}^{TT}_{1;\mu\nu} \tilde{h}_2^{TT'\mu\nu} - 2\tilde{\xi}'_{1;\mu}(g^{(R)\mu\nu} D^2 + \bar{R}^{\mu\nu})\tilde{\xi}'_{2;\nu} - 2\tilde{\xi}'_{1;\mu}\bar{R}^{\mu\nu} D_\nu \tilde{\sigma}'_2 \\ & \quad + \tilde{\sigma}'_1 \left(\frac{3}{4}(D^2)^2 + D_\mu \bar{R}^{\mu\nu} D_\nu\right)\tilde{\sigma}'_2 + \frac{1}{4}\tilde{\phi}_1\tilde{\phi}_2 \Big] \end{aligned}$$

zero modes $D_\mu \tilde{\xi}_\nu + D_\nu \tilde{\xi}_\mu = 0$ and $D_\mu D_\nu \tilde{\sigma} - \frac{1}{4} D^2 \tilde{\sigma} = 0$ are removed.

$$\Omega^2 = 1 + \phi.$$

For maximally symmetric spaces $\bar{R}^{\mu\nu} = C g^{(R)\mu\nu}$

$$\tilde{\xi} = \sqrt{-D^2 - C}\tilde{\xi}', \quad \tilde{\sigma} = \sqrt{(D^2)^2 + \frac{4}{3} C D^2}\tilde{\sigma}'$$

$$\langle h_1|h_2\rangle = \int dx\sqrt{g}\left[h^{TT}_{1;\mu\nu}h_2^{TT'\mu\nu} + 2\xi_1^{\mu}\xi_{2;\mu} + \frac{3}{4}\sigma_1\sigma_2 + \frac{1}{4}\phi_1\phi_2\right].$$

ghost fields:

$c_\mu = c^T_\mu + D_\mu(-D^2)^{-\frac{1}{2}}\rho,\ \bar{c}_\mu = \bar{c}^T_\mu + D_\mu(-D^2)^{-\frac{1}{2}}\bar{\rho},\ D_\mu c^{T\mu} = 0.$

$$\begin{aligned} S^{(2)}[g^{(R)}+h] + S_{gf}[g^{(R)}+h] \ =\ & \kappa_B^2\int dx\sqrt{g}\left[\frac{1}{2}h^{TT}_{\mu\nu}\left(-D^2 + \frac{2R}{3} - 2\lambda_B\right)h^{TT\mu\nu}\right. \\ & +\xi_\mu\left(-D^2 + \frac{R}{4} - 2\lambda_B\right)\xi^\mu + \frac{3}{8}\sigma(-D^2 - 2\lambda_B)\sigma \\ & \left.-\frac{1}{6}\phi(-D^2 - 2\lambda_B)\phi\right] \end{aligned}$$

$$S_{gh}[g^{(R)}, h=0, c^T, \bar{c}^T, \rho, \bar{\rho}] = -\sqrt{2}\kappa_B^2\int dx\sqrt{g}\left[\bar{c}^T_\mu\left(D^2 + \frac{R}{4}\right)c^{T\mu} + \bar{\rho}\left(D^2 + \frac{R}{4}\right)\rho\right]$$

D. Spherical harmonics

$$D^2 t^{(s)}_{\ell m} = -R\lambda^{(s)}_\ell t^{(s)}_{\ell m},$$

$$\begin{aligned} \lambda^{(s)}_\ell &= \frac{\ell(\ell+3)-s}{12}, \\ D^{(s)}_\ell &= \begin{cases} \frac{(2\ell+3)(\ell+1)(\ell+2)}{6} & s = 0(LL, Tr), \\ \frac{(2\ell+3)\ell(\ell+3)}{2} & s = 1(LT), \\ \frac{5(\ell+4)(\ell-1)(2\ell+3)}{6} & s = 2(TT). \end{cases} \end{aligned}$$

$$\Phi = \sum_{\ell=\ell^\Phi_{min}}^{\ell^\Phi_{max}}\sum_{m=1}^{D^\Phi_\ell} c^\Phi_{\ell m}t^\Phi_{\ell m}.$$

$$\langle t_1|t_2\rangle = \int dx\sqrt{g}\,t_1^* t_2,$$

$$\gamma^{(1)}(R) = \frac{D_\ell^{(2)}}{2}\sum_{\ell=2}^{\infty}\ln\left[\frac{\kappa_B^2 R}{\Lambda^4}\left(\lambda_\ell^{(2)}+\frac{2}{3}-\frac{2\lambda_B}{R}\right)\right]+\frac{D_\ell^{(1)}}{2}\sum_{\ell=2}^{\infty}\ln\left[\frac{\kappa_B^2 R}{\Lambda^4}\left(2\lambda_\ell^{(1)}+\frac{1}{2}-\frac{4\lambda_B}{R}\right)\right]$$
$$+\frac{D_\ell^{(0)}}{2}\sum_{\ell=2}^{\infty}\ln\left[\frac{3\kappa_B^2 R}{4\Lambda^4}\left(\lambda_\ell^{(0)}-\frac{2\lambda_B}{R}\right)\right]-D_\ell^{(1)}\sum_{\ell=1}^{\infty}\ln\left[\frac{\kappa_B^2 R}{\Lambda^4}\left(\lambda_\ell^{(1)}+\frac{1}{4}\right)\right]$$
$$-D_\ell^{(0)}\sum_{\ell=0}^{\infty}\ln\left[\frac{\kappa_B^2 R}{\Lambda^4}\left(\lambda_\ell^{(0)}+\frac{1}{2}\right)\right],$$

II. NUMERICAL RESULTS

A. Effective potential

$$\gamma^{(1)ninv}(R) = C'\ln\frac{\kappa_B^2 R}{\Lambda^4},$$

$$\gamma^{(1)}(R) = \begin{cases} M^4(\Lambda)\left(\frac{1}{R^2}-\frac{1}{R^2(\Lambda)}\right) & R \le R(\Lambda) \\ 0 & R > R(\Lambda) \end{cases}$$

$$M^4(\Lambda) = c_1\Lambda^4\ln\frac{c_2\kappa_B^2}{\Lambda^2},$$

$R(\Lambda) = c_3\Lambda^2$. $c_1 \approx 7.201$, $c_2 \approx 2.989$, $c_3 \approx 0.665$.

$$\gamma(R) = -\frac{v\kappa_B^2}{R}+c_1\Lambda^4\left(\frac{1}{R^2}-\frac{1}{R^2(\Lambda)}\right)\ln\frac{c_2\kappa_B^2}{\Lambda^2},$$

$$V = \int dx\sqrt{g} = \frac{v}{R^2}$$

$v = 3200\pi^2/3$.

Quantum phase transition at $\kappa_B^2 = \kappa_{cr}^2 = \Lambda^2/c_2$ when the curvature where $\gamma(R)$ reaches its minimum changes in discontinuous manner:

In the small cutoff phase, $\kappa_{cr}^2 << \kappa_B^2$, energetically favored curvature:

$$R_{min} = \frac{2c_1\Lambda^4}{v\kappa_B^2} \ln \frac{c_2\kappa_B^2}{\Lambda^2}$$

For large cutoff, $\kappa_B^2 < \kappa_{cr}^2$, $R_{min} = 0$.

B. Running Newton constant

$$\Gamma[g] = \int dx \kappa^2(R(x))\sqrt{g(x)}R(x),$$

$$\kappa^2(R) = \kappa_B^2 - \frac{R}{v}\gamma^{(1)}(R).$$

$$G(R) = \frac{1}{16\pi\kappa^2(R)} = \frac{1}{16\pi\kappa_B^2} \frac{1}{1 - \frac{c_1\Lambda^4}{v\kappa_B^2}\left(\frac{1}{R} - \frac{R}{R^2(\Lambda)}\right)\ln \frac{c_2\kappa_B^2}{\Lambda^2}}$$

perturbative for

$$\ell^2 > \ell_{Pl}^2(R) = \frac{\ell_B^2}{1 + \frac{16\pi c_1}{v}\Lambda^4\ell_B^2\left(\frac{1}{R} - \frac{R}{R^2(\Lambda)}\right)\ln \frac{16\pi\ell_B^2\Lambda^2}{c_2}},$$

whole theory is perturbative if $\ell_{Pl}\Lambda < 1$,

$$\Lambda^2\ell_B^2 < 1 + \frac{16\pi c_1}{v}\Lambda^4\ell_B^2\left(\frac{1}{R} - \frac{R}{R^2(\Lambda)}\right)\ln \frac{16\pi\ell_B^2\Lambda^2}{c_2}.$$

IR Landau pole in the small cutoff phase at $R_L = R_{min}/2$

All modes non perturbative and the theory strongly coupled unless $R > R_L$.

Modes perturbative as R increased beyond the Landau pole, entire theory perturbative for $R_{min} \leq R$.

Running Newton constant increasing function of R in the large cutoff phase

$$G(R) \approx \frac{Rv}{16\pi c_1\Lambda^4 \ln \frac{16\pi\ell_B^2\Lambda^2}{c_2}}$$

All modes perturbative for $R << \Lambda^2$.

C. Induced cosmological constant

$$\Gamma[g] = \kappa_{eff}^2 \int dx \sqrt{g(x)} F(R(x)),$$

$$\frac{dF(R)}{dR}_{|R=0} = 1$$

$F(R) = R - 2\lambda - gR^2$, $\kappa_{eff}^2 = \kappa_B^2$,

$$\lambda = \frac{c_1 \Lambda^4}{2v\kappa_B^2} \ln \frac{c_2 \kappa_B^2}{\Lambda^2}$$

$$g = -\frac{c_1 \Lambda^2}{2c_3 v\kappa_B^2} \ln \frac{c_2 \kappa_B^2}{\Lambda^2}$$

for $R << \Lambda^2$.

The induced cosmological constant positive/negative in the small/large cutoff phase; higher order derivative term generated with coupling of the opposite sign as the cosmological constant.

De Sitter background unstable in one-loop effective theory for $\lambda > cR$

$$c = \min\left(\frac{\lambda_\ell^2}{2} + \frac{1}{3}, \frac{\lambda_\ell^1}{2} + \frac{1}{8}, \frac{\lambda_\ell^0}{2}\right) = \frac{1}{2}.$$

$1/(\kappa_B \sqrt{-D^2 - 2\lambda})$ measure of amplitude of quantum fluctuations; qualitative claims for strength of interactions recovered.

III. CASIMIR EFFECT IN A BOX

$$e^{W[j]} = \int D[\phi] e^{\frac{\mu^2}{2}\langle\phi|\Box|\phi\rangle + \langle j|\phi\rangle},$$

$$\Gamma[\phi] = -\frac{\mu^2}{2}\langle\phi|\Box|\phi\rangle + \frac{1}{2}\mathrm{T}r \ln \frac{\mu^2}{2}\Box$$

$$\Gamma[0] = \frac{1}{2}\mathrm{T}r \ln \frac{\mu^2}{2}\Box$$

$$\tilde{\phi} = \sum_n c_n \frac{a^2}{L^2} e^{i\frac{2\pi}{L}n_\mu x^\mu}.$$

$$0 < n^2 \frac{(2\pi)^2}{L^2} < \Lambda^2 = \left(\frac{2\pi}{a}\right)^2$$

$$\Gamma[0] = \frac{1}{2}\sum_{n^\mu \neq 0} \Theta\left(\Lambda^2 - n^2\frac{(2\pi)^2}{L^2}\right)\left[-\ln\Lambda^4 + \ln\left(\mu^2 n^2 \frac{(2\pi)^6}{L^2}\right)\right]$$

For $1 << \Lambda L$

$$\begin{aligned}\Gamma[0] &= L^4 \int_{|p|\leq\Lambda} \frac{d^4p}{(2\pi)^4} \ln \frac{p^2\mu^2(2\pi)^4}{\Lambda^4} \\ &= \frac{L^4\Lambda^4}{32\pi^2}\left(\ln\frac{\mu^2(2\pi)^4}{\Lambda^2} - \frac{1}{2}\right).\end{aligned}$$

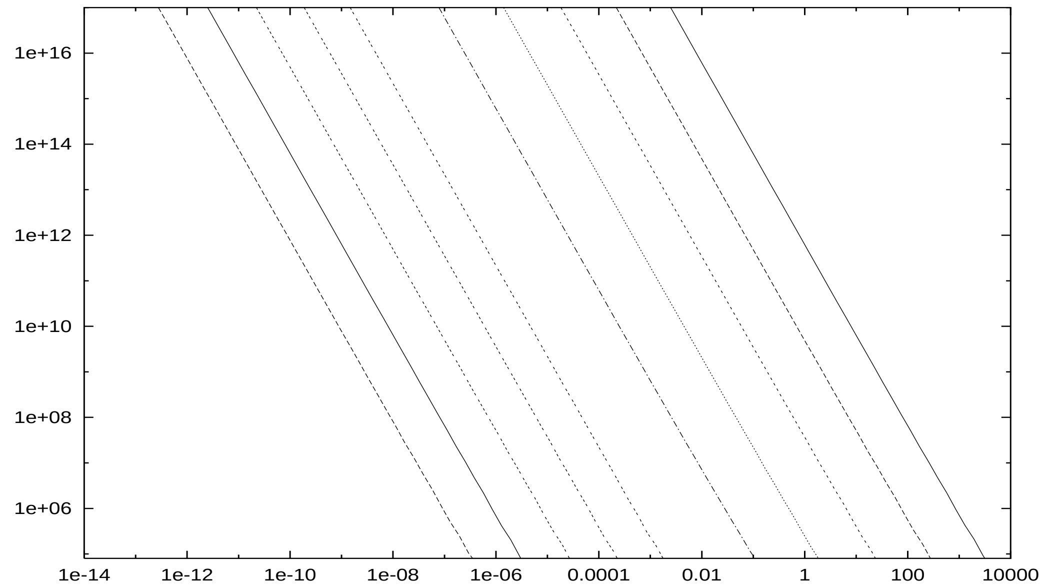

FIG. 1: The effective potential $|\gamma(R)|$ as a function of R/κ_B^2, obtained with sharp cutoff. The lines belong from left to right to $\Lambda^2/\kappa_B^2 = 10^{-5}$, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 1, 10, 10^2, 10^3, 10^4 and 10^5. Since $\gamma(R) < 0$ in the given range of curvature for $\Lambda^2/\kappa_B^2 \geq 1$ it is $-\gamma(R)$ which is shown for these values of the cutoff.

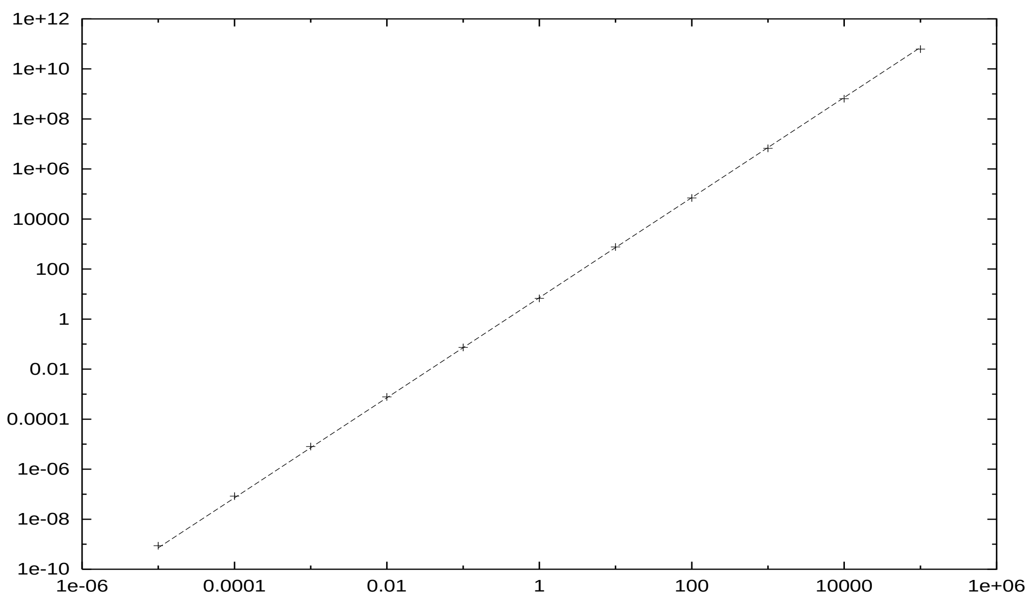

FIG. 2: The numerical and the fitted values for $M^4(\Lambda)/\ln c_2\kappa_B^2/\Lambda^2$ are shown by dots and continuous line, respectively, as functions of Λ^2/κ_B^2.